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Thermodynamics of Spacetime: The Einstein Equation of State

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Abstract

The Einstein equation is derived from the form of black hole entropy together with the fundamental relation $\delta Q = TdS$ connecting heat, entropy, and temperature. The key idea is to demand that this relation hold for all the local Rindler horizons through each spacetime point. Viewed in this way, the Einstein equation is an equation of state. It is born in the thermodynamic limit as a relation between thermodynamic variables, and its validity is seen to depend on the existence of local equilibrium conditions. As such there is no reason to think the gravitational field equations should be quantized, i.e., promoted to operator relations.

The four laws of black hole mechanics, analogous to those of thermodynamics, were originally derived from the classical Einstein equation[1]. With the discovery of the quantum Hawking radiation[2], it became clear that the analogy is in fact an identity. How did classical General Relativity know that horizon area would turn out to be a form of entropy, and that surface gravity is a temperature? In this essay I will answer that question by turning the logic around and deriving the Einstein equation from the form of black hole entropy together with the fundamental relation $\delta Q = TdS$ connecting heat Q, entropy S, and temperature T. Viewed in this way, the Einstein equation is an equation of state. It is born in the thermodynamic limit as a relation between thermodynamic variables, and its validity is seen to depend on the existence of local equilibrium conditions. As such there is no reason to think the gravitational field equations should be quantized, i.e., promoted to operator relations.

What plays the role of "heat" in gravitational dynamics? In thermodynamics, heat is energy exchanged among degrees of freedom that are not macroscopically observable. In gravidynamics, we can define heat as energy that flows across a horizon. It can be felt via the gravitational field it generates, but its particular form or nature is undetectable from outside the horizon. For the purposes of this definition it is not necessary that the "horizon" be a black hole event horizon. It can be simply the boundary of the past of any set $\mathcal{O}$ (for "observer"). This sort of horizon is a null hypersurface (not necessarily smooth) and, assuming cosmic censorship, it is composed of generators which are null geodesic segments emanating backwards in time from the set $\mathcal{O}$. We can consider a kind of "local" gravitational thermodynamics associated with such causal horizons, where the "system" is the degrees of freedom beyond the horizon.

Of particular interest to us are the past horizons of small two dimensional spacelike plane elements $\mathcal{P}$ whose null normal congruence to one side (the "outside") has vanishing expansion at $\mathcal{P}$. We call this the "local Rindler horizon of $\mathcal{P}$". Such a horizon defines a system that is instantaneously stationary (in "local equilibrium") at $\mathcal{P}$. Through any spacetime point there are local Rindler horizons in all null directions, and it will be shown here that the validity of $\delta Q = TdS$ for all of them implies the Einstein equation.

That causal horizons should be associated with entropy is suggested by the observation that they hide information[3]. Furthermore, the fine scale information that is hidden resides mostly in correlations between vacuum fluctuations on either side of the horizon and, for horizons large compared to a fundamental cutoff length scale l_c, this defines an entanglement entropy proportional to the horizon area at leading order in l_c^{-2}[4]. (The effect of subleading dependence on curvature and other fields will be briefly mentioned later.) Note that the area is an extensive quantity, as one expects for entropy.

So far we have argued that heat flow corresponds to energy flux across a causal horizon, and entropy is proportional to the area of that horizon. It remains to identify the temperature. We shall take it to be the Unruh temperature associated with the acceleration of an observer hovering just outside the horizon. To make this a little more precise, note that in a small neighborhood of any spacelike 2-plane element $\mathcal{P}$ one has an approximately flat region of spacetime, with the usual Poincaré symmetries. In particular, there is an approximate Killing field χ^a generating boosts orthogonal to $\mathcal{P}$

and vanishing at $\mathcal{P}$. According to the Unruh effect[5], the Minkowski vacuum state of quantum fields—or any state at very short distances— is a thermal state with respect to the boost hamiltonian at temperature $T = \hbar\kappa/2\pi$, where κ is the acceleration of the Killing orbit on which the norm of χ is unity. Actually, there is a 1-parameter family of such boost Killing fields, related by a constant rescaling, so this temperature is in fact ill-defined. In the case of a static black hole horizon in asymptotically flat spacetime the scale of the Killing field is fixed by normalization at infinity, so that the corresponding conserved quantity would be the energy-at-infinity. In the present context, we are not assuming staticity, nor are we assuming there is a black hole present, so this normalization condition is unavailable and we must simply accept the ambiguity. However, the "heat flux" δQ will have the same ambiguity, because the boost energy is different for the different normalizations, so this will not prevent us from finding the equation of state.

We are now ready to derive the equation of state from the relation $\delta Q = TdS$. Consider a spacetime point p. For each two dimensional plane element $\mathcal{P}$ through p there is a local Rindler horizon. Consider any one such horizon. Let χ^a be an approximate local boost Killing field generating this horizon, with the direction of χ^a chosen to be future pointing to the "inside" past of $\mathcal{P}$. We assume first that all the heat flow across the horizon is (boost) energy carried by matter, so that there is no "gravitational wave" energy flux. This heat flux just to the past of $\mathcal{P}$ is given by

$$\delta Q = \int_{\mathcal{H}} T_{ab}\chi^a d\Sigma^b,$$

where T_{ab} is the matter energy-momentum tensor. (In keeping with the thermodynamic limit, I assume the quantum fluctuations in T_{ab} are negligible.) The integral is over the "inside" past horizon of $\mathcal{P}$. If k^a is the tangent vector to the horizon generators for an affine parameter λ that vanishes at $\mathcal{P}$ and is negative to the past of $\mathcal{P}$, then $\chi^a = -\kappa\lambda k^a$ and $d\Sigma^a = k^a d\lambda d\mathcal{A}$, where $d\mathcal{A}$ is the area element on a cross section of the horizon. Thus the heat flux can also be written as

$$\delta Q = -\kappa \int_{\mathcal{H}} \lambda\, T_{ab}k^a k^b \, d\lambda d\mathcal{A}. \tag{1}$$

Assume now that the entropy is proportional to the horizon area, $dS = \eta\delta\mathcal{A}$. The dimensional constant η is undetermined by anything we have said so far (although given a microscopic theory of spacetime structure one may someday be able to compute η in terms of a fundamental length scale.) The area variation is given by

$$\begin{aligned} \delta\mathcal{A} &= \int_{\mathcal{H}} \partial_\lambda(d\mathcal{A})d\lambda \\ &= \int_{\mathcal{H}} \theta\, d\lambda d\mathcal{A} \end{aligned}$$

where $\theta = \partial_\lambda \ln(d\mathcal{A})$ is the expansion of the horizon generators.

The Raychaudhuri equation applied to the null geodesic congruence generating the horizon yields

$$\frac{d\theta}{d\lambda} = -\frac{1}{2}\theta^2 - \sigma^2 - R_{ab}k^a k^b$$

where $\sigma^2 = \sigma^{ab}\sigma_{ab}$ is the square of the shear. Since the local Rindler horizon is instantaneously stationary at $\mathcal{P}$, θ vanishes at $\mathcal{P}$, and θ^2 is a higher order contribution that can be neglected compared with the other terms when integrating to find θ near $\mathcal{P}$. Further, for simplicity we are assuming there is no gravitational heat flux. Therefore the horizon congruence is shear-free, so the σ^2 term vanishes. (We could use the Raychaudhuri equation and the heat flow equation to *define* the otherwise undefined gravitational contribution to the heat flux, thus eliminating this simplifying assumption.) With these assumptions we can integrate the Raychaudhuri equation to find $\theta = -\lambda\, R_{ab}k^a k^b$ for sufficiently small λ. Substituting this into the equation for $\delta\mathcal{A}$ we thus find

$$\delta A = -\int_{\mathcal{H}} \lambda\, R_{ab}k^a k^b\, d\lambda d\mathcal{A}. \tag{2}$$

With the help of (1) and (2) we can now see that $\delta Q = TdS = (\hbar\eta/2\pi)\delta A$ can only be valid if $T_{ab}k^a k^b = (\hbar\eta/2\pi)\, R_{ab}k^a k^b$ for all null k^a, which implies that $(2\pi/\hbar\eta)T_{ab} = R_{ab} + fg_{ab}$ for some function f. Local conservation of energy and momentum implies that T_{ab} is divergence free and therefore, using the contracted Bianchi identity, that $f = -R/2 + \Lambda$ for some constant Λ. We thus deduce that the Einstein equation holds with undetermined values for Newton's constant and the cosmological constant:

$$R_{ab} - \frac{1}{2}Rg_{ab} + \Lambda g_{ab} = \frac{2\pi}{\hbar\eta}T_{ab}.$$

The constant of proportionality η between the entropy and the area thus determines Newton's constant. The cosmological constant remains as enigmatic as ever in this analysis. Changing the assumed entropy functional would change the implied gravitational field equations. For instance, if the entropy density is given by a polynomial in the Ricci scalar $\alpha_0 + \alpha_1 R + ...$, then $\delta Q = TdS$ will imply field equations arising from a Lagrangian polynomial in the Ricci scalar[6].

Our derivation of the Einstein equation of state presumed the existence of local equilibrium conditions. Given such conditions, we have a system of local partial differential equations that is time reversal invariant and whose solutions include propagating waves. One might think of these as analogous to sound in a liquid, propagating as adiabatic compression waves. For sufficiently high frequencies one knows that the local equilibrium condition breaks down, entropy increases, and sound no longer propagates in a time reversal invariant manner. Similarly, one might expect that sufficiently high frequency or short wavelength disturbances of the gravitational field would no longer be described by the Einstein equation, not because some quantum operator nature of the metric would become relevant, but because the local equilibrium condition would fail. It is my hope that, by following this line of inquiry, we shall eventually reach an understanding of the nature of "non-equilibrium" gravitation.

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